Exam, pre-course in math and statistics

Krisztina Molnar Erik Ø. Sørensen*

September 5th 2011, 0900–1200, aud 22

You are allowed to use a calculator and one sheet of paper with notes. The calculators will be inspected before the exam starts and should not be micro-scale computers.

You should do prob/stat and the math parts on separate collections of paper, such that they can be graded independently.

In order to pass the exam, the answers to both the prob/stat and the math part must be acceptable on their own.

1 Probability and statistics (50%)

1. 15p: Let X and Y be independent random variables with uniform distribution on $(0,1)$. Calculate
   (a) $E(X)$,
   (b) $\text{var}(X)$,
   (c) $E(XY)$,
   (d) $E(\max(X,Y))$.
2. 10p: A point Y is chosen at random from a uniform distribution on $(0,1)$. A second point X is then chosen uniformly from the interval $(0,Y)$. Find the density for X.
3. 10p: If the density of (X,Y) is $f(x,y) = \exp(-(x+y))$ for $0 < x < \infty$ and $0 < y < \infty$, are X and Y independent?
4. 15p: Let X and Y have the joint pdf $f_{X,Y}(x,y) = 2\exp(-(x+y))$ for $0 < x < y < \infty$, zero elsewhere. Find the conditional mean $E(Y|X = x)$.

*Available at his cell phone, 93 44 22 89, if necessary.

5. 15p: The random variables X_1 and X_2 are independent with means and variances μ_1, μ_2 and σ_1^2, σ_2^2 respectively. Let $Y = X_1 X_2$.

   (a) Calculate $\text{cov}(Y, X_1)$

   (b) Are Y and X_1 independent?

6. 15p: A true-false examination has 48 questions. June has probability $3/4$ of answering a question correctly. April just guesses on each question. A passing score is 30 or more correct answers. Compare the probability that June passes the exam with the probability that April passes it. Give a brief verbal argument for any approximations you make.

7. 20p: Åshild, a worker for the Department of Fish and Game, is assigned the job of estimating the number of trout in a certain lake of modest size. She proceeds as follows: She catches 100 trout, tags them, and puts them back in the lake. One month later, she catches 100 more trout, and notes that 10 of them have tags.

   (a) Without doing any fancy calculation, give a rough estimate of the number of trout in the lake.

   (b) Let N be the number of trout in the lake. Find an expression, in terms of N, for the probability that Åshild would catch 10 tagged trout out of the 100 trout that she caught the first time. Assume that there is no change in the population in that month.

   (c) Find the value of N which maximizes the expression in part (b). What is such an estimator for N called?

   (d) A manager believes that there are at most 800 trout in the lake. If there were 800 trouts in the lake, how many would Åshild expect to catch with a tag the second time?

   (e) Should Åshild attempt to convince her manager that he is wrong?

2 Mathematics (50%)

You should explain/prove all your answers - no points without it!

1. Is the following statement true: $\{\sim (A \vee B)\} \Leftrightarrow \{(\sim A) \wedge (\sim B)\}$?

2. Let $A = \{1,2,3,4\}$ and $B = \{\alpha, \beta, \gamma, \delta\}$. Is the following relation a function?
$$\mathscr{R} = \{(1,\gamma),(2,\delta),(3,\delta),(4,\beta)\}$$

3. Do the set of integers and even integers have the same cardinality?

4. Let $f(x) = 1/x$. Does $\lim_{x\to 0} f'(x)$ exist?

5. Are the following functions injective (one to one) and surjective (onto)?

   (a) $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$,

   (b) $f: \mathbb{R} \to [0,\infty), \quad f(x) = x^2$,

   (c) $f: [0,\infty) \to \mathbb{R}, \quad f(x) = x^2$,

   (d) $f: [0,\infty) \to [0,\infty), \quad f(x) = x^2$.

6. Calculate the following integrals,

   (a)
   $$\int (2x-1)e^x dx,$$

   (b)
   $$\int_0^1 (1-x)^2 dx,$$

   (c)
   $$\int \frac{x-2}{x(x-4)} dx.$$

7. Solve the following differential equation, you should get the implicit equation of ellipses. Is there any restriction on the constant of integration?
$$\frac{dy}{dx} = -\frac{x}{2y}.$$

8. Prove the following inequality without calculating the integrals,
$$\int_0^{10} \frac{x}{1+x} dx \leq \int_0^{10} \ln(1+x) dx.$$

9. Prove that
$$\lim_{v\to 1}\frac{1-v^2}{1-v}=2.$$

10. Calculate the length of vector $\boldsymbol{a}$,
$$\boldsymbol{a}=\begin{pmatrix}3\\2\\1\end{pmatrix}.$$

11. What is the set of points where $x+2y+3z=0$? (Possible answers: empty set, a point, a line, a plane, a sphere, none of the above.)

12. Let us consider matrix $\boldsymbol{A}$, that makes a 90 degree rotation,
$$\boldsymbol{A}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$
What are the eigenvectors of $\boldsymbol{A}$, $\boldsymbol{x}_1=(x_{11},x_{22}),\boldsymbol{x}_2=(x_{21},x_{22})$ such that $x_{i,j}\in\mathbb{R}$ for $i,j=1,2$?

13. For the following function, find the critical points, and classify them using the second derivative criterion,
$$f(x,y)=3x^2+xy+y^2-x-2y+4.$$

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997