

Exam, pre-course in math and statistics

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August 27th 2012, 0900–1200, aud 22.

You are allowed to use a calculator and one sheet of paper with notes. The calculators will be inspected before the exam starts and should not be micro-scale computers.

You should do prob/stat and the math parts on separate collections of paper, such that they can be graded independently.

In order to pass the exam, the answers to both the prob/stat and the math part must be acceptable on their own.

1 Probability and statistics (50%)

1. (15p) For each of these statements, indicate whether they are true or false. Give only a brief justification.
 - (a) Let A be an event in the sample space \mathcal{C} , with A^c as its complement. If $P(A) = P(A^c)$, then $P(A) = 1/2$.
 - (b) Let $(\mathcal{C}, \mathcal{B}, P)$ be a probability space, with \mathcal{C} the sample space, \mathcal{B} a σ -field on \mathcal{C} and P the probability set function on \mathcal{B} . If $A \in \mathcal{B}$, then $(A \cup A^c)^c \in \mathcal{B}$.
 - (c) For any two events A and B in the σ -field \mathcal{B} defined on the sample space \mathcal{C} , $P(A \cup B) = P(A) + P(B)$.
 - (d) Under certain conditions, it is possible that the sum of probabilities of all the sample points in a sample space is less than one.
2. (10p) Let X be uniformly distributed on $(0, 1)$. Find the density of $Y = X^2$.
3. (10p) Let X be distributed with cumulative distribution function F_X . Let $Y = F_X(X)$. What is the distribution of Y ?

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4. (15p) Suppose X has the pdf

$$f_X(x) = \begin{cases} ax^2 & \text{if } 0 < x < 3, \\ 0 & \text{elsewhere.} \end{cases}$$

- (a) Determine a .
- (b) Calculate $E(X)$, $\text{var}(X)$, and the median of X .
- (c) In a random sample of size n , with n large, what is the probability that the sample average, $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$, is larger than $E(X)$?
- (d) In a random sample of size $n = 73$, calculate an approximation to the probability that $\bar{X} > 2$.

5. (15p) Let X be distributed $\Gamma(k, \theta)$, such that it has a moment generating function

$$m(t) = (1 - t\theta)^{-k}.$$

- (a) Calculate $E(X)$ and $\text{var}(X)$.
- (b) Let Z be the decentered variable, $Z = X - E(X)$. What is the moment generating function of Z ?

6. (20p) Let X and Y be independent exponential variables with densities

$$f(s) = \begin{cases} \lambda e^{-\lambda s} & \text{for } s \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

- (a) Calculate the moment generating function for X , $m(t) = E(e^{tX})$.
- (b) Calculate $E(X)$ and $\text{var}(X)$.
- (c) Define $Z = X + Y$. Find the moment generating function of Z and calculate $E(Z)$ and $\text{var}(Z)$.

7. (15p) Define f as

$$f(x, y) = \begin{cases} x + y & \text{for } x, y \in [0, 1], \\ 0 & \text{elsewhere.} \end{cases}$$

- (a) Show that f is a joint probability density function for X and Y .
- (b) Are X and Y independent?
- (c) Calculate the marginal density of X .
- (d) Calculate the density of Y conditional on X .
- (e) Calculate $E(Y|X)$.

2 Math (50%)

You should explain/prove all your answers - no points without it!

1. (3p) Define function f , with domain \mathbb{R} and range $\{x \in \mathbb{R} : x \geq 0\}$ by the formula $f(x) = x^2$. Does this function have an inverse?
2. (5p) Consider the following statement

$$-x^2 + 5x - 4 > 0 \Rightarrow x > 0.$$

- a) Use direct proof to prove the above statement.
 - b) Use indirect proof to prove the above statement.
3. (3p) Prove the following statement

$$1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$$

4. (3p) Calculate the following limit

$$\lim_{x \rightarrow 0} \frac{x^2 - x}{x^3 + x^2 + x}$$

5. (3p) Are the following functions monotone

$$f(x) = \frac{1}{1+x}, \quad x \geq 0$$
$$f(x) = \frac{1}{1+x}, \quad x \neq -1$$

6. (4p) Let λ be an eigenvalue of matrix A , and x be the corresponding (non-zero) eigenvector. Let I denote the identity matrix, which is the same size as A . Explain why the determinant of $(A - \lambda I)$ is zero.
7. (4p) Are the following two vectors linearly dependent? $v_1 = (0, 0)$ and $v_2 = (0, 1)$.
8. (4p) Calculate the area under the parabola $f(x) = x^2$ over the interval $[0, 1]$.
9. (5p) Calculate the area between the x -axis and the graph of $f(x) = e^{x/3} - 3$ over the interval $[0, 3 \ln 3]$. (Note that $f(3 \ln 3) = 0$.)

10. (5p) Show that if

$$\lim_{n \rightarrow \infty} a_n = q > 1, \Rightarrow \lim_{n \rightarrow \infty} a_n^n = \infty.$$

(Use the definition of convergence of a sequence!)

11. (5p) Consider the following optimization problem:

$$\begin{aligned} \max_{x,y} f(x,y) &= \alpha x^2 + y^2; \\ \text{s.t. } x + y &= 1 \end{aligned}$$

Does the optimal value of f increase if α increases?

12. (7p) Let

$$f(x) = |\ln|x||^{x^2}.$$

Determine $\lim_{x \rightarrow 0} f(x)$.

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997