Exam, pre-course in math and statistics

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You are allowed to use a calculator and one sheet of paper with notes. The calculators will be inspected before the exam starts and should not be micro-scale computers.

You should do prob/stat and the math parts on separate collections of paper, such that they can be graded independently.

In order to pass the exam, the answers to both the prob/stat and the math part must be acceptable on their own.

Enclosed at the end is a standard normal distribution table that might be useful to you.

1 Probability and statistics (50%)

1. 5p: Which (if any) of these collections are potentially σ-fields over some sample space such that probability functions could be defined over them? Explain briefly.

   (a) $\mathscr{C}_a = \{\varnothing, \{A, B, C\}, \{A\}, \{B\}, \{C\}\}$

   (b) $\mathscr{C}_b = \{\varnothing, \{A, B, C\}, \{A\}, \{B, C\}\}$

   (c) $\mathscr{C}_c = \{\varnothing, \{1, 2, 3\}, \{4, 5\}\}$

2. 10p: Suppose X has the pdf

$$f_X(x) = \begin{cases} ax^2 & \text{if } 0 < x < 3, \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Determine a.

   (b) Calculate $E[X]$, $\text{var}[X]$, and the median of X.

   (c) Given a random sample of size $n = 73$, approximate the probability that the average is larger than 2.

3. 7p: Let X be uniformly distributed on $(0, 1)$. Find the density of $Y = X^2$.

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4. 8p: Let the joint density of X and Y be

$$f_{XY}(x,y) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Are X and Y independent?

   (b) Define $Z = X^2 + Y^2$. Calculate $E[Z]$.

5. 12p: Let X be an exponential random variable with density

$$f_X(x) = \lambda e^{-\lambda x}, \quad \text{for } x \geq 0.$$

   (a) Calculate the moment-generating function of X, $m_X(t)$.

   (b) For what values of t is the $m_X(t)$ you've found defined?

   (c) Let Y be a random variable such that X and Y are iid. Calculate the moment-generating function for $Z = X + Y$.

6. 8p: You have estimated a parameter θ, and have both an estimate, $\hat{\theta}$, and its standard error, $\widehat{\sigma}_{\hat{\theta}}$. Provide an estimate of

$$\xi = \frac{e^{\theta}}{1 + e^{\theta}},$$

   and a large-sample approximation of its standard error.

2 Mathematics (50%)

1. 5p: Is matrix A positive definite?

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

   Is the inverse of A positive definite? Explain your answers.

2. 5p: Prove that if A and B square matrixes are positive definite, their sum $A + B$ is also positive definite.

3. 5p: Calculate the following integrals

$$\int e^{x^2} x \; \mathrm{d}x \tag{a}$$

$$\int_0^1 x^2 + 1 \; \mathrm{d}x \tag{b}$$

4. 5p: Solve the following differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}.$$

5. 5p: A projection is a linear transformation $\boldsymbol{P}$ from a vector space to itself such that $\boldsymbol{P}^2 = \boldsymbol{P}$. That is, whenever P is applied twice to any value, it gives the same result as if it were applied once. What numbers can be eigenvalues of a projection matrix $\boldsymbol{P}$?

6. 5p: Examine the following functions. Do they have extreme values and if yes what type? Determine the intervals on which the functions are monotone.

$$f(x) = x^4 - x^2 \tag{a}$$

$$f(x) = \frac{x}{x^2 + 1} \tag{b}$$

7. 5p: Calculate the following limit

$$\lim_{x \to 0} \frac{x^2}{x^3 + x^2 + x}.$$

8. 5p: Prove the following statement

$$1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n + 1).$$

9. 10p: Under what parameter values of a, b is the following Cobb Douglas function concave?

$$u(x, y) = x^a y^b$$

(Recall, a function is convex if the Hessian is positive definite.)

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997