Exam, PhD pre-course in math and statistics

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You are allowed to use a calculator and one sheet of paper with notes. The calculators will be inspected before the exam starts and should not be micro-scale computers.

You should do prob/stat and the math parts on separate collections of paper, such that they can be graded independently.

In order to pass the exam, the answers to both the prob/stat and the math part must be acceptable on their own.

Enclosed at the end is a standard normal distribution table that might be useful to you.

1 Mathematics (50%)

1. (5p) Prove by induction the formula $1+3+5+\cdots+(2n-1)=n^2$, which is valid for all $n=1,2,3,\ldots$.

2. (5p) Let A be a finite set with a elements, and let B be a finite set with b elements. How many different functions $f : A \to B$ are there? How many functions are there if A and/or B is replaced by the empty set $\emptyset$?

3. (5p) Integrate by parts
$$\int \frac{1}{x-x^2}\,\mathrm{d}x.$$

4. (5p) Solve the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y - y^2$$
with the initial condition $y(0) = 1/3$.

5. (5p) Show that
$$\lim_{x\to 0}\left(1-\frac{1}{x}\right)^x = \frac{1}{e}.$$

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6. (5p) Determine whether the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are linearly independent or not:

   (a)
   $$\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \boldsymbol{c} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

   (b)
   $$\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad \boldsymbol{c} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}.$$

7. (10p) Let
   $$\boldsymbol{A} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}.$$
   Find the eigenvalues λ_1 and λ_2 of $\boldsymbol{A}$, and find corresponding eigenvectors. What is
   $$\boldsymbol{A}^{100} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$
   (Hint: Try to write the vector $(2,0)'$ as the sum of two eigenvectors of $\boldsymbol{A}$.)

8. (10p) Consider the problem to maximize $f(x,y) = xy + y$ given the conditions $x + y \leq R$, $x \geq 0$, $y \geq 0$, where $R > 0$ is a parameter.

   (a) Compute the gradient vector $\nabla f(x,y)$. For which values of $R > 0$ is the point $(x,y) = (0,R)$ a possible candidate for the solution to the maximizing problem?

   (b) Find the solution $x = x(R)$, $y = y(R)$ for all $R > 0$ to the maximizing problem above.

   (c) Set up the value function $V = V(R)$ of the problem. What can you say about $V'(R)$?

2 Probability and statistics (50%)

1. (7p) For the following cdfs $F(s)$, find the pdf, the median, and the 25th percentile.

   (a) $F(s) = 1/(1+e^{-s})$ for $-\infty \leq s \leq \infty$.

   (b) $F(s) = \exp(-e^{-s})$ for $-\infty \leq s \leq \infty$.

2. (8p) Let X have the pdf $f(x) = 3x^2$ for $0 < x < 1$, zero elsewhere.

   (a) Compute $\mathrm{E}[X^3]$.

   (b) Let $Y = X^3$. Show that it has a uniform distribution on $(0,1)$.

(c) Consider a rectangle with sides X and $(1-X)$. What is the expected area of this rectangle?

3. (10p) Let X_1 and X_2 have the joint pdf

$$f(x_1,x_2) = \begin{cases} 6x_2 & 0 < x_2 < x_1 < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Calculate the marginal pdf of X_1.

(b) Calculate the conditional pdf of X_2, given $X_1 = x_1$.

(c) Calculate $\mathrm{E}[X_2|X_1]$. Is this a random variable?

4. (5p) Determine the correlation coefficient of the random variables X and Y if $\mathrm{var}[X] = 4$, $\mathrm{var}[Y] = 2$, and $\mathrm{var}[X+2Y] = 15$.

5. (10p) Define f as

$$f(x,y) = \begin{cases} x+y & \text{for } x,y \in [0,1], \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Show that f is a joint probability density function for X and Y.

(b) Are X and Y independent?

(c) Calculate the marginal density of X.

(d) Calculate the density of Y conditional on X.

(e) Calculate $E[Y|X]$.

6. (5p) In the population, IQ scores are normally distributed with an average 100 and a standard deviation 15.

(a) What percent of the people have IQ scores under 70?

(b) What IQ score corresponds to the 84th percentile?

(c) In a random sample of 79 NHH students, the average IQ score is 107. What is the p-value for the hypothesis that NHH students are drawn randomly from the population?

7. (5p) Let $\{X_n\}$ be a sequence of random variables, with $\operatorname{plim} X_n = s$. Let

$$g(t) = \begin{cases} 0 & \text{if } t < s-2 \\ t & \text{for } s-2 \le t \le s+2 \\ s+2 & \text{if } t > s+2. \end{cases}$$

Is it possible to find $\operatorname{plim} g(X_n)$?

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997