# Exam, PhD pre-course in math and statistics

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You are allowed to use a calculator and one sheet of paper with notes. The calculators will be inspected before the exam starts and should not be micro-scale computers.

You should do prob/stat and the math parts on separate collections of paper, such that they can be graded independently.

In order to pass the exam, the answers to both the prob/stat and the math part must be acceptable on their own.

Enclosed at the end is a standard normal distribution table that might be useful to you.

## 1 Mathematics (50%)

1. (10p) The propositions $p_1, p_2, \dots, p_n$ are said to logically imply the proposition $q$ if $(p_1 \land p_2 \land \cdots \land p_n) \Rightarrow q$ is a tautology. Let $p, q, r$ be the propositions

   $p$: Roger studies.
   $q$: Roger plays tennis.
   $r$: Roger passes his math exam.

   Let $p_1, p_2, p_3$ be the propositions

   $p_1$: $p \Rightarrow r$
   $p_2$: $\neg q \Rightarrow p$
   $p_3$: $\neg r$

   Do the propositions $p_1, p_2, p_3$ logically imply the proposition $q$? Explain.

2. (10p) Use integration by parts to show the formula

$$\int x^n e^{-x}\,dx = -x^n e^{-x} + n\int x^{n-1} e^{-x}\,dx.$$

   Use induction to prove the formula

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

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for all $n = 0, 1, 2, \ldots$. *Note*: The *factorial* is defined by $0! = 1$ and $n! = 1 \cdot 2 \cdots n$ for $n = 1, 2, 3 \ldots$.

3. (10p) Solve the differential equation

$$P'(t) + 2P(t) = 3, \qquad P(0) = 1.$$

   Determine $\lim_{t \to \infty} P(t)$.

4. (10p) Determine whether the vectors $u_1, u_2, u_3$ are linearly independent or not.

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}.$$

   What is the dimension of the subspace $L \subset \mathbf{R}^3$ spanned by the vectors $u_1, u_2, u_3$? *Note*: $L$ is given by $L = \{c_1 u_1 + c_2 u_2 + c_3 u_3 \,:\, c_1, c_2, c_3 \in \mathbf{R}\}$.

5. (15p) Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Find all eigenvalues and corresponding eigenvectors belonging to the matrix $A$. Prove by induction that $A^n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for all $n = 1, 2, \ldots$, where $A^n$ is the $n$-fold product $A^n = A \cdot \cdots \cdot A$.

6. (15p) Consider the following optimization problem: Maximize $f(x, y) = xy$ given the constraints $g_1(x, y) = x + 2y \leq 6$, $g_2(x, y) = x \leq k$, $x \geq 0$, and $y \geq 0$. (It will not be necessary to consider the constraints $x \geq 0$ and $y \geq 0$ in this problem. They are only included to ensure that a solution to the problem exists, and you may disregard them in your solution.) Here $k$ is a constant, and we assume that $0 < k < 6$. For which values of $k$ does the point $(x, y) = \left(k, \frac{6-k}{2}\right)$ satisfy the Karush-Kuhn-Tucker conditions?

## 2 Probability and statistics (50%)

1. (10p) Assume that there are bags of tulip bulbs in the basement, and that they contain 25 bulbs each. Yellow bags contain 20 yellow tulips and 5 red tulips, and red bags contain 15 red and 10 yellow tulips. 60% of the bags in the basement are yellow, the others are red. One bulb is chosen at random from a random bag in the basement, and then planted.

   (a) What is the probability that the tulip turns out yellow?

   (b) Given that the tulip turns out yellow, what is the probability that it came from a yellow bag?

2. (15p) Consider random variables $X$ and $Y$ with $E[X^2] < \infty$ and $E[Y^2] < \infty$. For each of these statements indicate whether they are true or false.

(a) $E[X] < \infty$ and $E[Y] < \infty$.

(b) $\text{var}[X]$ need not exist.

(c) $\text{cov}[X, Y] = 0$ implies that $E[X|Y] = E[X]$.

(d) $X$ and $Y$ are independent implies that $E[X|Y] = E[X]$.

(e) $X$ and $Y$ are independent is equivalent to $E[X|Y] = E[X]$.

(f) $E[X|Y] = E[X]$ implies that $E[Y|X] = E[Y]$.

(g) Both $E[X|Y]$ and $E[Y|X]$ are random variables.

3. (10p) If the density of $(X, Y)$ is $f(x, y) = \exp(-(x + y))$ for $0 < x < \infty$ and $0 < y < \infty$, are $X$ and $Y$ independent?

4. (20p) Let $X$ and $Y$ have joint density

$$f(x, y) = \begin{cases} 21x^2y^3 & \text{for } 0 < x < y < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density of $Y$, $f_Y(y)$, remember to pay careful attention to the areas of positive density.

(b) Find the conditional density of $X$, $f_{X|Y}(x|y)$.

(c) Find $E[X|Y]$.

(d) Find $E\left[E[X|Y]\right]$.

5. (25p) Let $X$ have probability density function $f(x) = \theta x^{\theta-1}$ defined on $(0, 1)$ and zero elsewhere.

(a) Calculate $E[X]$ and $\text{var}[X]$.

(b) Assume $\theta = 3$ and that you get a sample of size $n = 100$ on $X$. Can you give an approximate expression for the distribution of $\overline{X} = \sum_{i=1}^{n} X_i/n$?

(c) Approximate the probability that $\overline{X}$ is within one percentage point of $E[X]$, $P(|\overline{X} - \mu| < 0.01)$. You can use the enclosed table.

**STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.**

| Z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | .50000 | .50399 | .50798 | .51197 | .51595 | .51994 | .52392 | .52790 | .53188 | .53586 |
| **0.1** | .53983 | .54380 | .54776 | .55172 | .55567 | .55962 | .56356 | .56749 | .57142 | .57535 |
| **0.2** | .57926 | .58317 | .58706 | .59095 | .59483 | .59871 | .60257 | .60642 | .61026 | .61409 |
| **0.3** | .61791 | .62172 | .62552 | .62930 | .63307 | .63683 | .64058 | .64431 | .64803 | .65173 |
| **0.4** | .65542 | .65910 | .66276 | .66640 | .67003 | .67364 | .67724 | .68082 | .68439 | .68793 |
| **0.5** | .69146 | .69497 | .69847 | .70194 | .70540 | .70884 | .71226 | .71566 | .71904 | .72240 |
| **0.6** | .72575 | .72907 | .73237 | .73565 | .73891 | .74215 | .74537 | .74857 | .75175 | .75490 |
| **0.7** | .75804 | .76115 | .76424 | .76730 | .77035 | .77337 | .77637 | .77935 | .78230 | .78524 |
| **0.8** | .78814 | .79103 | .79389 | .79673 | .79955 | .80234 | .80511 | .80785 | .81057 | .81327 |
| **0.9** | .81594 | .81859 | .82121 | .82381 | .82639 | .82894 | .83147 | .83398 | .83646 | .83891 |
| **1.0** | .84134 | .84375 | .84614 | .84849 | .85083 | .85314 | .85543 | .85769 | .85993 | .86214 |
| **1.1** | .86433 | .86650 | .86864 | .87076 | .87286 | .87493 | .87698 | .87900 | .88100 | .88298 |
| **1.2** | .88493 | .88686 | .88877 | .89065 | .89251 | .89435 | .89617 | .89796 | .89973 | .90147 |
| **1.3** | .90320 | .90490 | .90658 | .90824 | .90988 | .91149 | .91309 | .91466 | .91621 | .91774 |
| **1.4** | .91924 | .92073 | .92220 | .92364 | .92507 | .92647 | .92785 | .92922 | .93056 | .93189 |
| **1.5** | .93319 | .93448 | .93574 | .93699 | .93822 | .93943 | .94062 | .94179 | .94295 | .94408 |
| **1.6** | .94520 | .94630 | .94738 | .94845 | .94950 | .95053 | .95154 | .95254 | .95352 | .95449 |
| **1.7** | .95543 | .95637 | .95728 | .95818 | .95907 | .95994 | .96080 | .96164 | .96246 | .96327 |
| **1.8** | .96407 | .96485 | .96562 | .96638 | .96712 | .96784 | .96856 | .96926 | .96995 | .97062 |
| **1.9** | .97128 | .97193 | .97257 | .97320 | .97381 | .97441 | .97500 | .97558 | .97615 | .97670 |
| **2.0** | .97725 | .97778 | .97831 | .97882 | .97932 | .97982 | .98030 | .98077 | .98124 | .98169 |
| **2.1** | .98214 | .98257 | .98300 | .98341 | .98382 | .98422 | .98461 | .98500 | .98537 | .98574 |
| **2.2** | .98610 | .98645 | .98679 | .98713 | .98745 | .98778 | .98809 | .98840 | .98870 | .98899 |
| **2.3** | .98928 | .98956 | .98983 | .99010 | .99036 | .99061 | .99086 | .99111 | .99134 | .99158 |
| **2.4** | .99180 | .99202 | .99224 | .99245 | .99266 | .99286 | .99305 | .99324 | .99343 | .99361 |
| **2.5** | .99379 | .99396 | .99413 | .99430 | .99446 | .99461 | .99477 | .99492 | .99506 | .99520 |
| **2.6** | .99534 | .99547 | .99560 | .99573 | .99585 | .99598 | .99609 | .99621 | .99632 | .99643 |
| **2.7** | .99653 | .99664 | .99674 | .99683 | .99693 | .99702 | .99711 | .99720 | .99728 | .99736 |
| **2.8** | .99744 | .99752 | .99760 | .99767 | .99774 | .99781 | .99788 | .99795 | .99801 | .99807 |
| **2.9** | .99813 | .99819 | .99825 | .99831 | .99836 | .99841 | .99846 | .99851 | .99856 | .99861 |
| **3.0** | .99865 | .99869 | .99874 | .99878 | .99882 | .99886 | .99889 | .99893 | .99896 | .99900 |
| **3.1** | .99903 | .99906 | .99910 | .99913 | .99916 | .99918 | .99921 | .99924 | .99926 | .99929 |
| **3.2** | .99931 | .99934 | .99936 | .99938 | .99940 | .99942 | .99944 | .99946 | .99948 | .99950 |
| **3.3** | .99952 | .99953 | .99955 | .99957 | .99958 | .99960 | .99961 | .99962 | .99964 | .99965 |
| **3.4** | .99966 | .99968 | .99969 | .99970 | .99971 | .99972 | .99973 | .99974 | .99975 | .99976 |
| **3.5** | .99977 | .99978 | .99978 | .99979 | .99980 | .99981 | .99981 | .99982 | .99983 | .99983 |
| **3.6** | .99984 | .99985 | .99985 | .99986 | .99986 | .99987 | .99987 | .99988 | .99988 | .99989 |
| **3.7** | .99989 | .99990 | .99990 | .99990 | .99991 | .99991 | .99992 | .99992 | .99992 | .99992 |
| **3.8** | .99993 | .99993 | .99993 | .99994 | .99994 | .99994 | .99994 | .99995 | .99995 | .99995 |
| **3.9** | .99995 | .99995 | .99996 | .99996 | .99996 | .99996 | .99996 | .99996 | .99997 | .99997 |