Exam, PhD pre-course in math and statistics

Krisztina Molnar and Erik Ø. Sørensen*

September 4, 2017, 0900–1200, "midt-i-mellom I"

You are allowed to use a calculator and one sheet of paper with notes. The calculators will be inspected before the exam starts and should not be micro-scale computers.

You should do prob/stat and the math parts on separate collections of paper, such that they can be graded independently.

In order to pass the exam, the answers to both the prob/stat and the math part must be acceptable on their own.

Enclosed at the end is a standard normal distribution table that might be useful to you.

1 Mathematics (50%)

1. Check monotonicity, boundedness and convergence of the following sequences

   (a) $a_n = \frac{n^2+1}{n^2+n}$

   (b) $b_n = (-1)^n \frac{1}{2n^2}$

2. Calculate the limit of the following sequences using

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad \text{and} \quad \lim_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^n = e^\alpha$$

   (a) $a_n = \left(1+\frac{1}{n}\right)^{n+1}$

   (b) $b_n = \left(1-\frac{1}{n}\right)^n$

3. Let

$$f(x) = 4 - 5x + 2x^3 - x^5.$$

   Is it true, that

$$f'(a) = f'(-a)?$$

*Available at cell phone 93 44 22 89.

4. Examine the following function, does it have minima or maxima and where?

$$f(x, y) = (3 - 2x + y)e^{-y^2} \tag{1}$$

5. For what values of a, b is the following function concave?

$$U(x, y) = x^a y^b$$

Hint, in other words, when is $-u(x, y)$ convex? You can show convexity by writing up a quadratic formula with a second order Taylor approximation and checking whether the Hessian is positive definite. Recall, that a symmetric matrix is positive definite if all the eigenvalues are positive. You can find eigenvelues λ_1, λ_2 of a 2 by 2 matrix as $\lambda_1 + \lambda_2 = a_{11} + a_{22}$ (Trace) and $\lambda_1 \lambda_2 = a_{11}a_{22} - a_{12}a_{21}$ (Determinant), where a_{ij} denotes the matrix element in row i column j.

6. Consider the matrix of a 90° rotation. Does it have real eigenvectors $v = (v_1, v_2), v \in \mathbb{R}$? (Explain your answer.)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

2 Probability and statistics (50%)

1. Consider a single throw of an ordinary six-sided dice.
   (a) Define the elements of a corresponding probability space $(S, \mathscr{A}, P)$.
   (b) Consider the function X defined on this space,

   $$X(s) = \begin{cases} 0 & \text{if } s \in \{\{1\}, \{2\}, \{3\}\}, \\ 1 & \text{if } s \in \{\{4\}, \{5\}, \{6\}\}. \end{cases}$$

   What is the $\sigma(X)$, the σ-algebra generated by X?
   (c) How can we find the probability space for X?

2. Let X be a random variable that is uniformly distributed on (a, b).
   (a) Calculate $E[X]$ and $\text{var}[X]$.
   (b) Calculate $E[e^{tX}]$, the moment generating function of X.

3. Based on these joint probability densities, determine if the random variables X and Y are independent:
   (a) The joint pdf of the random variables X and Y is $f(x, y) = 12xy(1 - y^2)$ for $0 < x < 1$ and $0 < y < 1$ and zero elsewhere.

(b) The joint pdf of the random variables X and Y is $g(x, y) = 3x$ for $0 < y < x < 1$ and zero elsewhere.

4. Determine the mean and variance of the average $\overline{X}_n$ of a random sample of size n from a distribution with pdf $f(x) = 4x^3$ for $0 < x < 1$ and zero elsewhere.

5. Let X_1 and X_2 be independent with normal distributions $N(6, 1)$ and $N(7, 1)$, respectively. Find $P(X_1 > X_2)$.[1]

6. A common blood test indicates the presence of a disease 95% of the time when the disease is actually present in an individual. Joe's doctor draws some of Joe's blood, and performs the test on his drawn blood. The results indicate that the disease is present in Joe.

   Here's the information that Joe's doctor knows about the disease and the diagnostic blood test:

   - One-percent (that is, 1 in 100) people have the disease. That is, if D is the event that a randomly selected individual has the disease, then $P(D) = 0.01$.
   - If H is the event that a randomly selected individual is disease-free, that is, healthy, then $P(H) = 1 - P(D) = 0.99$.
   - The "sensitivity" of the test is 0.95. That is, if a person has the disease, then the probability that the diagnostic blood test comes back positive is 0.95. That is, $P(T^+|D) = 0.95$.
   - The "specificity" of the test is 0.95. That is, if a person is free of the disease, then the probability that the diagnostic test comes back negative is 0.95. That is, $P(T^-|H) = 0.95$.
   - If a person is free of the disease, then the probability that the diagnostic test comes back positive is $1 - P(T^-|H) = 0.05$. That is, $P(T^+|H) = 0.05$.

   What is the positive predictive value of the test? That is, given that the blood test is positive for the disease, what is the probability that Joe actually has the disease?

[1] Hint: Define a new variable $Y = X_1 - X_2$.

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997